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On Herstein's Theorems Relating Jordan and Associative Algebras

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In ([3]; also [5]) I. N. Herstein gave a construction associating with any Jordan ideal J in a Jordan algebra of the form A^+ an associative ideal B in the associative algebra A. He used this to show that if A was simple, so was A^+. In ([4]; also [5]) he gave an analogous construction associating with any Jordan ideal J in the Jordan algebra $H(A, *)$ of $*$-symmetric elements an associative $*$-ideal B in the associative algebra A with involution $*$. This was used to show (in characteristic $\neq 2$) that if A was $*$-simple then $H(A, *)$ was simple.

The purpose of this paper is to show that his constructions can also be used to relate the radicals of A^+ or $H(A, *)$ with that of A. Specifically we will prove

THEOREM 1. *If A is an associative algebra then the Jacobson radical* Rad A *of A coincides with the Jacobson radical* Rad A^+ *of the Jordan algebra A^+,* Rad $A =$ Rad A^+.

THEOREM 2. *If A is an associative algebra then the nil radical $N(A)$ of A coincides with the nil radical $N(A^+)$ of the Jordan algebra A^+, $N(A) = N(A^+)$.*

THEOREM 3. *If A is an associative algebra with involution $*$ then the Jacobson radical of the Jordan algebra $H(A, *)$ of symmetric elements is* Rad $H(A, *) = H(A, *) \cap$ Rad A.

We will also extend Professor Herstein's simplicity theorems to the characteristic 2 case.

1. PRELIMINARIES

Throughout this note A denotes an associative algebra (not necessarily unital) over a commutative associative ring Φ of scalars. (If A is a ring,

Φ is just the integers, while for linear algebras it is a field). For convenience in forming principal ideals we will frequently make use of the algebra

$$A' = \Phi 1 + A$$

obtained by adjoining a unit to A (in case it doesn't already have one). An involution on A extends in a natural way to one on A', and similarly an ideal (or Jordan ideal) in A remains one in A'.

We are using the notion of quadratic Jordan algebra introduced in [*1*]; the fundamental operations are a cubic composition $U_x y$ and a quadratic composition x^2. If A is an associative algebra we get a Jordan algebra, denoted A^+, by taking

$$U_x y = xyx, \qquad x^2 = xx.$$

The linearized compositions are denoted

$$U_{x,z} y = \{x\, y\, z\} = xyz + zyx, \qquad x \circ z = xz + zx.$$

Any Jordan subalgebra of an algebra A^+ (i.e. a subspace closed under $U_x y$ and x^2) is called a *special* Jordan algebra. If A has an involution $*$ then the subspace $H(A, *)$ of $*$-symmetric elements forms a special Jordan algebra.

An *ideal* in a Jordan algebra J is a subspace K such that $U_J K$, $U_K J$, K^2, $K \circ J$ are all contained in K. More succinctly, using the algebra J' obtained by adjoining a unit, the conditions amount to

$$U_{J'} K \subset K, \qquad U_K J' \subset K. \tag{1}$$

If $J \subset A^+$ is special this means all xzx, zxz, z^2, $zx + xz$ belong to K for $x \in J$, $z \in K$. If K is an ideal in J then any (Jordan) product involving at least one factor from K will fall back in K; for example, $\{z\, x\, y\} \in K$ if $x, y \in J$ and $z \in K$.

In a special Jordan algebra, an element x has a (Jordan) *inverse* y if and only if y is the (associative) inverse of x. Similarly, x is *quasi-invertible* (or quasi-regular) with quasi-inverse y in the special Jordan algebra if and only if it is in the associative algebra, i.e. if $1 - x$ has inverse $1 - y$ (in A'). We will use q.i. as an abbreviation of "quasi-invertible"; a subspace is q.i. if all its elements are q.i. The *Jacobson radical* of an associative or Jordan algebra is the maximal q.i. ideal, and the *nil radical* is the maximal nil ideal (see [*2*]).

2. The First Construction

This construction involves the relation between Jordan ideals in A^+ and associative ideals in A. Note that any associative ideal is necessarily a Jordan ideal, but not conversely.

THE FIRST HERSTEIN CONSTRUCTION. Let A be an associative algebra and J a Jordan ideal in A^+. Then

(i) If $b^2 \neq 0$ for some $b \in J$ then the principal ideal $B = A'b^2A'$ generated by b^2 is a nonzero associative ideal of A contained in J.

(ii) If $b^2 = 0$ for all $b \in J$ then for any $b \neq 0$ in J the principal ideal $B = A'bA'$ generated by b is a nonzero associative ideal of A (not necessarily contained in J) with $B^3 = 0$.

COROLLARY. *If A is a semiprime associative algebra then every nonzero Jordan ideal J of A^+ contains a nonzero associative ideal B of A.*

THEOREM 4. *A is simple as an associative algebra if and only if A^+ is simple as a Jordan algebra.*

Proof of the construction: we first claim that for any $b \in J$

$$bA'bA' \subset J \tag{2}$$

$$A'b^2A' \subset J. \tag{3}$$

The first follows from

$$\begin{aligned} bxby &= bx(by) + (by)\,xb - b(yx)\,b \\ &= \{b\;x\;by\} + U_b(yx) \subset \{J\;A'\;A'\} + U_JA' \subset J, \end{aligned}$$

so the second follows from

$$\begin{aligned} xb^2y = (xb + bx)(by) + (by)(bx + xb) - bxby - bybx \\ - b(yx)\,b \in (A' \circ J) \circ A' - bA'bA' - bA'bA' - U_JA' \subset J. \end{aligned}$$

Part (i) is just (3); for part (ii), if $b^2 = 0$ for all $b \in J$ then $ab + ba = 0$ for all $a, b \in J$ so that $bA'bA'b = (bA'bA')\,b = -b(bA'bA') = 0$ by (2) since $b^2 = 0$. Clearly this implies $B = A'bA'$ has $B^3 = 0$.

3. PROOF OF THEOREMS 1 AND 2

Since Rad A (meaning the Jacobson radical or the nil radical respectively) is an associative ideal in A it is certainly also a Jordan ideal in A^+, whose elements are q.i. (or nilpotent) in A^+ since they are q.i. (or nilpotent) in A, so

$$\text{Rad } A \subset \text{Rad } A^+.$$

The hard part is the other direction. It will be enough if the image $\overline{\text{Rad } A^+}$ of Rad A^+ in $\bar{A} = A/\text{Rad } A$ is zero. Now $\overline{\text{Rad } A^+} \subset \text{Rad } \bar{A}^+$ since the homomorphic image of a q.i. (or nil) ideal is again a q.i. (or nil) ideal, so it will be enough if Rad $\bar{A}^+ = 0$. Thus we have reduced Theorem 1 (or 2) to

THEOREM 1′. *B is semisimple as an associative algebra if and only if* B^+ *is semisimple as a Jordan algebra.*

THEOREM 2′. *B is free of nil (associative) ideals if and only if* B^+ *is free of nil (Jordan) ideals.*

Proof. We have just seen one direction, B^+ semisimple (or free of nil ideals) implies B semisimple (or free of nil ideals). The other way, if B is semisimple (or free of nil ideals) then $\operatorname{Rad} B^+$ is a Jordan ideal in the semisimple (or nil-free), hence semiprime, associative algebra B. By the Corollary to the First Herstein Construction, if $\operatorname{Rad} B^+$ were nonzero it would contain a nonzero associative ideal C. Then all elements of C are q.i. (or nilpotent) since the elements of $\operatorname{Rad} B^+$ are. But since B is semisimple (or nil-free) it can't contain any nonzero q.i. (or nil) ideals, so $\operatorname{Rad} B^+$ must be zero.

This proof works quite generally. Suppose we have some category of algebras, and some property P of such algebras. Say P is a *radical property* if

(i) homomorphic images of P-algebras are P-algebras

(ii) any ideal in a P-algebra is a P-algebra

(iii) if B is an ideal in A such that B and A/B are P-algebras, then A is a P-algebra.

The *P-radical* $R_P(A)$ is the maximal P-ideal (if such exists); when it exists one has $R_P(A/R_P(A)) = 0$. Any finite sum of P-ideals is again a P-ideal; say that P is *summable* if an arbitrary sum of P-ideals is a P-ideal. In this case the P-radical always exists (as the sum of all P-ideals). In our case, suppose that P is a summable radical property for associative algebras and P^+ a summable radical property for Jordan algebras which "correspond" in the sense that an associative algebra A has P if and only if the Jordan algebra A^+ has P^+. Assume also that P is strong enough to include nilpotent ideals: if $A^3 = 0$ then A is a P-algebra. Then the proof of Theorems 1 and 2 carries over, mutatus mutandus, to show that the P-radical of A coincides with the P^+-radical of A^+,

$$R_P(A) = R_{P^+}(A^+).$$

4. THE SECOND CONSTRUCTION

For the second construction we consider an associative algebra A with involution * and relate the *-ideals (i.e. *-invariant ideals) in the associative algebra A to Jordan ideals in the Jordan algebra $H = H(A, *)$. If B is a $*$-ideal it turns out that the *kernel*

$$K(B) = \left\{ b + b^* + \sum \lambda_i b_i b_i^* + \sum b_j h_j b_j^* \text{ for } b, b_i, b_j \in B, h_j \in H, \lambda_i \in \Phi \right\}$$

of the ideal B plays a more important role than the space

$$H(B) = H(B, *) = H \cap B$$

of all $*$-symmetric elements in B. Note that $K(B) \subset H(B)$ since B is a $*$-ideal. If $\frac{1}{2} \in \Phi$ (for example, if Φ is a field of characteristic $\neq 2$) then $K(B) = H(B)$ since any $b = b^*$ in $H(B)$ may be written $b = \frac{1}{2}(b + b^*) \in K(B)$.

We claim that $K(B)$ is a Jordan ideal in H if B is a $*$-ideal in A. Recall the definition (1). Clearly $U_{K(B)}H' \subset K(B)$ since we even have $bH'b^* \subset K(B)$ for $b \in B$, and $U_{H'}K(B) \subset K(B)$ since $aK(B)\, a^* \subset K(B)$ for $a \in A'$: $a(b + b^*)\, a^* = (aba) + (aba)^* = b'' + b''^*$ and $a(bh'b^*)\, a^* = (ab)\, h'(ab)^* = b'h'b'^*$ for $b' = ab \in B$, $b'' = aba \in B$.

Of special interest is the ideal $K = K(A)$, the *kernel* of A. In well-behaved algebras we have $K = H$. Indeed, we will have $K = H$ if any of the following hold:

(i) $\frac{1}{2} \in \Phi$

(ii) A has a unit 1

(iii) $*$ is an involution of the second kind

(iv) $*$ is the exchange involution on the direct sum $A = B \oplus B^*$ of anti-isomorphic algebras B, B^*.

We have already seen that if $\frac{1}{2} \in \Phi$ then $K = K(A) = H(A) = H$. For (ii), if $1 \in A$ then any $h = 1h1^*$ for $h \in H$ belongs to K. Both (iii) and (iv) follow since in these cases every symmetric element has the form $a + a^*$ for $a \in A$—in case (iii) because there is a λ in the centroid with $\lambda + \lambda^* = 1$, so $h = \lambda h + \lambda^* h = (\lambda h) + (\lambda h)^*$ for any $h \in H$, and in (iv) because any symmetric element has the form $b \oplus b^*$. If $K \neq H$ we are in trouble, since any subspace lying between K and H is an ideal:

$$\text{If } K \subset J \subset H \text{ then } J \text{ is an ideal in } H. \tag{4}$$

Referring to (1), note that $U_J H + U_H J \subset U_H H \subset K$ since any $aHa^* \subset K$, $J^2 \subset H^2 \subset K$ since any $aa^* \in K$, and $J \circ H \subset K$ since for $z \in J$, $h \in H$ we have $z \circ h = zh + hz = (zh) + (zh)^* \in K$.

We will have occasion to consider objects more general than ideals. We say a subspace $J \subset H$ is a *kernel ideal* if it is a Jordan subalgebra such that (in analogy with (1))

$$U_J K \subset J, \qquad U_K J \subset J, \qquad J \circ K \subset J.$$

Any Jordan ideal of H or any Jordan ideal of K is automatically a kernel ideal.

In an algebra A with involution $*$ a subspace L of symmetric elements is called *ample* if it contains all norms aa^* and all traces $a + a^*$ and if $aLa^* \subset L$ for all $a \in A$. The largest ample subspace is the space H itself, and the smallest is the *core* of A, the space $K_0(A)$ spanned by the norms and traces. For any $*$-ideal B in A we can define the *core* of B to be

$$K_0(B) = \left\{b + b^* + \sum \lambda_i b_i b_i^* \text{ for } b, b_i \in B, \lambda_i \in \Phi\right\}.$$

Clearly $K_0(B) \subset K(B)$, and the previous arguments show that $K_0(B) = K(B) = H(B)$ in cases (i), (iii), (iv). The *ample hull* $A(S)$ of a subspace $S \subset H$ is the smallest ample subspace $K_0(A) + S + \sum_{a \in A} aSa^*$ containing S. We say $J \subset H$ is a *core ideal* if it is an ideal in its ample hull; this is equivalent to the condition that J be a subalgebra with

$$U_{K_0} J \subset J \qquad U_J K_0 \subset J \qquad J \circ K_0 \subset J \qquad U_J(aJa^*) \subset J$$
$$U_{aJa^*} J \subset J \qquad J \circ aJa^* \subset J \qquad \{J\, K_0\, aJa^*\} \subset J$$
$$\{aJa^*\ J\ bJb^*\} \subset J$$

for all $a, b \in A$. Any Jordan ideal or kernel ideal of H is a core ideal, as is any Jordan ideal of K_0.

We now show that if $J \subset H$ is a Jordan ideal there is an associative $*$-ideal B whose kernel $K(B)$ is contained in J.

THE SECOND HERSTEIN CONSTRUCTION. If A is an associative algebra with involution $*$ and J is an ideal or kernel ideal in $H(A, *)$ then

(i) If $bcb \neq 0$ for some b, c in J the principal ideal $B = A'bcbA'$ generated by bcb is a nonzero associative $*$-ideal in A with $K(B) \subset J$

(ii) If $bcb = 0$ for all b, c in J but $J \neq 0$ then A contains a nonzero nilpotent ideal.

If J is merely a core ideal the same conclusions hold except that in (i) we can only conclude $K_0(B) \subset J$.

COROLLARY. *If A is a semiprime algebra with involution then for any nonzero ideal or kernel ideal J in $H(A, *)$ there is a nonzero associative $*$-ideal B in A with $K(B) \subset J$. If J is merely a core ideal then we can only conclude $K_0(B) \subset J$.*

THEOREM 5. *Let A be a *-simple associative algebra with involution. Then the kernel $K(A)$ is a simple Jordan algebra. The Jordan ideals in $H(A)$ are precisely all subspaces J lying between $K(A)$ and $H(A)$, $K \subset J \subset H$, so $H(A, *)$ is a simple Jordan algebra if and only if $H(A) = K(A)$. The core $K_0(A)$ is also a simple Jordan algebra.*

The Corollary follows immediately from the Construction. To obtain the Theorem, let J be any nonzero kernel ideal in H (resp. core ideal in K_0). Since A is *-simple it is semiprime, so by the Corollary there is a nonzero *-ideal B in A with $K(B) \subset J$ (resp. $K_0(B) \subset J$). Since A is *-simple, the only nonzero *-ideal is $B = A$, so $K(A) \subset J$ (resp. $K_0(A) \subset J$). This immediately shows $K(A)$ (resp. $K_0(A)$) is simple, since any nonzero ideal J in $K(A)$ (resp. $K_0(A)$) is a kernel ideal (resp. core ideal). It also shows $K(A) \subset J$ for any nonzero ideal in H, so by (4) the ideals in H are precisely the $K \subset J \subset H$.

Thus for *-simple A, $K(A)$ is always simple, though $H(A)$ may not be. No instance where $K(A) \neq H(A)$ for *-simple A is known to the author, but it would not be surprising if such existed. It is easy to give examples where $K_0(A) \neq K(A)$: let A be the algebra Ω_n of $n \times n$ matrices over a field Ω of characteristic 2, $\Phi = \mathbb{Z}_2$ the prime field, * the transpose. Then $K_0(A)$ is the set of all symmetric matrices whose diagonal entries lie in Ω^2, while $K(A) = H(A)$ is all symmetric matrices. Thus $K_0(A) = K(A)$ if and only if Ω is perfect.

Proof of the construction. Let $J \subset H$ be a kernel (resp. core) ideal. We first verify

$$(bkb)\, x + x^*(bkb) \in J \quad \text{if} \quad b \in J, \quad x \in A', \quad k \in K' \quad (\text{resp. } A(J)') \tag{5}$$

$$xbcbx^* \in J \quad \text{if} \quad b, c \in J \quad \text{and} \quad x \in A' \tag{6}$$

$$xbcby^* + ybcbx^* \in J \quad \text{if} \quad b, c \in J \quad \text{and} \quad x, y \in A'. \tag{7}$$

For (5),

$$\begin{aligned}(bkb)\, x + x^*(bkb) &= bk(bx + x^*b) + (bx + x^*b)\, kb - b(kx^* + xk)\, b \\ &= \{b\, k\, k_1\} - bk_2b \in J\end{aligned}$$

since $b \in J$ and $k_1 = (bx) + (bx)^*$, $k_2 = (xk) + (xk)^* \in K_0' \subset K'$. For (6),

$$\begin{aligned}xbcbx^* &= (xb + bx^*)\, c(xb + bx^*) - xb(cx + x^*c)\, b - b(cx + x^*c)\, bx^* \\ &\quad + (xbx^*)\, cb + bc(xbx^*) - b(x^*cx)\, b \\ &= k_1ck_1 - \{x(bk_2b) + (bk_2b)\, x^*\} + \{k_3\, c\, b\} - bk_4b \in J\end{aligned}$$

by (5) since $k_1 = (xb) + (xb)^*$, $k_2 = (cx) + (cx)^* \in K_0 \subset K$ and $k_3 = xbx^*$, $k_4 = x^*cx$ belong to K (resp. $A(J)$). Then (7) follows from (6) by linearization.

Consider $B = A'bcbA'$; since $b^* = b$, $c^* = c$ this is an associative $*$-ideal in A. Now B is spanned by all $xbcby$ for x, y in A', so $K(B)$ (resp. $K_0(B)$) is spanned by all $xbcby + y^*bcbx^*$, $(xbcby)\,h'(xbcby)^* = xbc'bx^*$ for $c' = cbyh'y^*bc = U_cU_b(yh'y^*)$, and $(xbcby)h'(zbcbw)^* + (zbcbw)h'(xbcby)^* = xbcby' + y'^*bcbx^*$ for $y' = yh'w^*bcbz^*$ where $h' \in H'$ (for $K_0(B)$ we take $h' = 1$). The first and third of these are in J by (7), and the second is in J by (6) since $c' \in J$: if J is a kernel ideal this follows since $yh'y^* \in K'$, and if J is a core ideal and $h' = 1$ then $yy^* \in K_0'$. Hence $K(B) \subset J$ (resp. $K_0(B) \subset J$), proving (i).

Finally, we come to (ii). Suppose $bcb = 0$ for all $b, c \in J$; in particular $b^3 = 0$ for all b. Then by (5) $[x(bkb) + (bkb)\,x^*]^3 = 0$ for all $x \in A'$, $b \in J$, $k \in K'$ (resp. $A(J)'$). If we multiply on the left by $x(bkb)$ we get

$$\begin{aligned} 0 &= x(bkb)[x(bkb) + (bkb)\,x^*]^3 \\ &= [x(bkb)]^4 \end{aligned}$$

since $x(bkb) \cdot (bkb)\,x^* = 0$ (by $bkb^2kb = bcb = 0$ where $c = kb^2k \in J$ if $k \in K'$ or $A(J)'$ resp.) and if $yz = 0$ then $y(y + z)^n = y^{n+1}$. Thus the left ideal $B = A'bkb$ is nil of index 4. Either some B is nonzero or else $bkb = 0$ for all $b \in J$, $k \in K'$ (resp. $A(J)'$). In the latter case, for arbitrary $x \in A'$ we have $k = x + x^* \in K_0' \subset K'$ so $bxb = -bx^*b$; but then $(xb)^3 = +xbxbxb = -xb(xbx^*)\,b = 0$ since $xbx^* \in K$ (resp. $A(J)$), and $A'b$ is a nil left ideal of index 3 which is nonzero if $b \neq 0$. In any case, if $J \neq 0$ then A contains some nonzero left ideal which is nil of index $\leqslant 4$. A theorem of Levitzki [see 5, p. 1] says that if A contains a nonzero one-sided nil ideal of bounded index it contains a nonzero nilpotent ideal. This establishes (ii).

5. Proof of Theorem 3

Just as in Theorems 1 and 2, the easy direction is $H \cap \operatorname{Rad} A \subset \operatorname{Rad} H$ ($H = H(A, *)$) since $H \cap \operatorname{Rad} A$ is a Jordan ideal in H whose elements are q.i. in H (any $z \in H \cap \operatorname{Rad} A$ has a quasi-inverse w *in* A; but then w^* is the quasi-inverse of $z^* = z$, so by the uniqueness of the inverse $w^* = w$ belongs to H, and z is q.i. *in* H). For the opposite direction it would suffice if $\overline{\operatorname{Rad} H} = 0$ in $\bar{A} = A/\operatorname{Rad} A$. Since $\operatorname{Rad} A$ is a $*$-ideal, $\bar{A}$ inherits an involution $\bar{*}$. Thus $\bar{A}$ is a semisimple algebra with involution, and one is tempted to try to reduce Theorem 3 to

> If B is a semisimple associative algebra with involution $*$ then $H(B, *)$ is a semisimple Jordan algebra.

(Note we don't say "if and only if"; given any algebra of characteristic $\neq 2$ with involution such that $H(A, *)$ is semisimple we can tack on a skew subspace M to A to obtain an algebra $B = A \oplus M$ with multiplication $(a + m)(a' + m') = aa'$ and involution $(a + m)^* = a^* - m$ so that $H(B, *) = H(A, *)$ is still semisimple but B has grown a radical M).

However, a slight hitch prevents us from reducing Theorem 3 to the above. As before we can argue that $\overline{\text{Rad}\, H}$ is a q.i. ideal in the homomorphic image $\bar{H} = \overline{H(A, *)}$, but to apply the above we would need it a q.i. ideal in $H(\bar{A}, \bar{*})$. Certainly $\overline{H(A, *)} \subset H(\bar{A}, \bar{*})$, but in general passage to $\bar{A}$ may introduce $\bar{*}$-symmetric elements which don't come from $*$-symmetric ones: the condition $\bar{a}^{\bar{*}} = \bar{a}$ is equivalent to $\overline{a^* - a} = 0$ and thus to $a^* = a + z$ for $z \in \text{Rad}\, A$. (This difficulty doesn't arise if $\frac{1}{2} \in \Phi$, since then any such $\bar{a}^{\bar{*}} = \bar{a}$ comes from the symmetric element $\frac{1}{2}(a + a^*)$ in A).

However, $\overline{\text{Rad}\, H}$ is a core ideal: it is an ideal in its ample hull since it is an ideal in $\bar{H}$, which is an ample subspace containing $\overline{\text{Rad}\, H}$ and hence the ample hull $\bar{A}(\overline{\text{Rad}\, H})$. Indeed, the core $K_0(\bar{A})$ is just the image $\overline{K_0(A)}$ of the core of A since both are spanned by the elements $\bar{a} + \bar{a}^{\bar{*}} = \overline{a + a^*}$, $\bar{a}\,\bar{a}^{\bar{*}} = \overline{aa^*}$ for $a \in A$. Clearly $\bar{H}$ contains the core $K_0(\bar{A})$ and $\overline{\text{Rad}\, H}$. It is ample since $\bar{a}\,\bar{H}\,\bar{a}^{\bar{*}} = \overline{aHa^*} \subset \bar{H}$. Thus $\overline{\text{Rad}\, H}$ is still a q.i. core ideal in $H(\bar{A}, \bar{*})$. Our arguments have shown that Theorem 3 will follow from the stronger result

THEOREM 3′. *If B is a semisimple associative algebra with involution $*$ then $H(B, *)$ is semisimple as a Jordan algebra; more generally, $H(B, *)$ contains no nonzero* q.i. *core ideals.*

This is really the only reason we bothered with core ideals in the Second Herstein Construction: we needed a notion of ideal which was weak enough to include $\overline{H(A, *)}$ and yet strong enough to carry through the construction.

Before we can prove Theorem 3′ we need a lemma.

LEMMA. *If A is an associative algebra with involution $*$ then an element b is* q.i. *if and only if both $b + b^* - bb^*$ and $b + b^* - b^*b$ are* q.i.

Proof. If b is q.i. so is b^*, and both $1 - b$ and $1 - b^*$ are invertible. Then $(1 - b)(1 - b^*) = 1 - k$ and $(1 - b^*)(1 - b) = 1 - h$ are invertible for $k = b + b^* - bb^*$, $h = b + b^* - b^*b$, so k and h are q.i. Conversely, if k and h are q.i. then both $(1 - b)(1 - b^*)$ and $(1 - b^*)(1 - b)$ are invertible, $1 - b$ has a right and a left inverse and is thus invertible, so b is q.i.

COROLLARY. *If B is a $*$-subalgebra of A such that $K_0(B)$ is* q.i. *then B is* q.i.

Proof of Theorem 3′. Suppose J were a nonzero core ideal in H. Since B is semisimple it is semiprime, so by the Corollary to the Second Construction if J were nonzero there would be a nonzero associative $*$-ideal C in B with

$K_0(C) \subset J$. But this is impossible, since then $K_0(C)$ would be q.i. and by the above corollary so would C, implying $C \subset \operatorname{Rad} B = 0$.

The proof we have given will again apply to any summable radical properties P and P^+ for associative algebras with involution and Jordan algebras if P and P^+ "correspond" in the sense that if A has P then $H(A)$ has P^+, and if $K_0(A)$ has P^+ then A has P, and if P is strong enough to include nilpotent ideals. In particular, we could show that the nil radical $N(H)$ of H was just $N(H) = H \cap N(A)$ if we had an analogue of the above Lemma or its Corollary for nilpotent elements.

QUESTION. Let B be an associative algebra with involution $*$ such that all $*$-symmetric elements are nilpotent. Is B itself necessary nil?

(Note that if every norm bb^* is nilpotent then so is every symmetric element h since $h^2 = hh^*$ is nilpotent). By the above Lemma such a B is necessarily a radical algebra. It is also not hard to show B contains nonzero nil onesided ideals. (We can find $z \neq 0$ in B with $zz^* = 0$ by taking $b \neq 0$ if $bb^* = 0$ or $(bb^*)^{n-1}$ if $(bb^*)^{n-1} \neq 0$ but $(bb^*)^n = 0$. Then $xz \in B$ for all $x \in A'$, so $xz + (xz)^*$ is nilpotent by hypothesis: $(xz + z^*x^*)^n = 0$. Then again $zz^* = 0$ gives $0 = xz(xz + z^*x^*)^n = (xz)^{n+1}$, so $A'z$ is a nonzero nil left ideal). Thus the question would receive an affirmative answer if the Koethe Conjecture were true.

6. ALTERNATE PROOFS

Recall that an element $z \in A$ is *properly q.i.* (p.q.i.) if za is q.i. for all $a \in A$. The radical consists precisely of the p.q.i. elements.

Alternate Proof of Theorem 1. We show directly that $\operatorname{Rad} A^+ \subset \operatorname{Rad} A$ by showing that all $z \in \operatorname{Rad} A^+$ are p.q.i. To prove za is q.i. it suffices if $(za)^2$ is q.i.; but by (2) with $b = z$, $J = \operatorname{Rad} A^+$ we see $zaza$ is in $\operatorname{Rad} A^+$ and hence certainly q.i.

Alternate Proof of Theorem 3. In case A has an involution we show directly that $\operatorname{Rad} H \subset \operatorname{Rad} A$ ($H = H(A, *)$) by showing that all $z \in \operatorname{Rad} H$ are p.q.i., again by verifying that $(za)^2$ is q.i. By the Lemma it will be enough if $(za)^2 + (za)^{2*} - (za)^2(za)^{2*}$ and $(za)^2 + (za)^{2*} - (za)^{2*}(za)^2$ are q.i. But

$$
\begin{aligned}
(za)^2 + (za)^{2*} &= zaza + a^*za^*z \\
&= z(a + a^*)\, za + a^*z(a + a^*)\, z - z(a^*za) - (a^*za)\, z \\
&= (zk_1z)\, a + a^*(zk_1z) - (zk_2 + k_2z)
\end{aligned}
$$

for $k_1 = a + a^*$, $k_2 = a^*za \in K$, and by (5) both $(zk_1z)\, a + a^*(zk_1z)$ and $zk_2 + k_2z$ belong to the ideal $\operatorname{Rad} H$ in H. Thus $(za)^2 + (za)^{2*} \in \operatorname{Rad} H$.

Therefore it will be enough if $(za)^2(za)^{2*}$ and $(za)^{2*}(za)^2$ are q.i., and one of these will be q.i. if and only if the other is. But $(za)^2(za)^{2*} = zazaa^*za^*z = zhz$ belongs to Rad H since z does and $h = azaa^*za^*$ belongs to H. Thus $(za)^2(za)^{2*}$ is q.i., which finishes the proof that z is p.q.i.

These proofs are a little more direct, but they cannot be applied to general radical properties P. For example, Theorem 2 cannot be proved in this way, since one doesn't know whether the nil radical consists precisely of the properly nilpotent elements (this is one form of the Koethe Conjecture).

Another Proof of Theorem 1. Theorem 1 is actually a special case of Theorem 3. If we take B to be the direct sum $A \oplus A^0$ of A with its opposite algebra and take $*$ to be the exchange involution then the map $a \to a \oplus a$ is an isomorphism $A^+ \to H(B, *)$, under which Rad A^+ corresponds to $H(B, *) \cap \text{Rad}\, B = H(B, *) \cap (\text{Rad}\, A \oplus \text{Rad}\, A) \cong (\text{Rad}\, A)^+$.

The same argument can be applied to show A^+ is simple when A is: in this case B is $*$-simple, and we have seen that $K(B) = H(B)$, so the simplicity of $K(B)$ implies that of $A^+ \cong H(B)$.

In general, results about A^+ are special cases of results about $H(A, *)$. Thus it is not surprising that results for $H(A, *)$ are harder to prove.

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